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MAT 312

ALUPE UNIVERSITY

OFFICE OF THE DEPUTY VICE CHANCELLOR

ACADEMICS, RESEARCH AND STUDENTS' AFFAIRS

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER REGULAR MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION ARTS/SCIENCE

COURSE CODE: MAT 312

COURSE TITLE: COMPLEX ANALYSIS I

DATE: 19th DECEMBER 2023 **TIME:** 2.00PM – 5.00PM

INSTRUCTION TO CANDIDATES

- SEE INSIDE

STREAM: BED (Arts/Science) **DURATION: 3 Hours**

INSTRUCTIONS TO CANDIDATES

i. *Answer ALL Questions from* ***section A and ANY from section B***.

ii. *Do not write on the question paper.*

SECTION A (31 Marks)

Answer ALL questions in this section.

QUESTION ONE (16 Marks)

a) Find all solutions of the complex number $z^2 = -5 + 12i$ and give your answer in the form $z = x + iy$. (4 Marks)

b) Differentiate the following complex functions from first principles:

i) $f(z) = z^2 + z$ (2 Marks)

ii) $f(z) = 1/z$ (3 Marks)

c) Write the function $f(z) = |z|$ in the form $u(x, y) + iv(x, y)$. Using the Cauchy-Riemann equations, decide whether there are any points in $\mathbb{C}$ at which f is differentiable. (4 Marks)

d) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n!}$. (3 Marks)

QUESTION TWO (15 Marks)

a) Let γ denote the circular path with centre 1 and radius 1, described once anticlockwise and starting at the point 2. Let $f(z) = |z|^2$. Write down a parametrisation of γ. Hence calculate $\int_{\gamma} |z|^2 dz$. (6 Marks)

b) Find the Taylor expansion of $\sin^2 z$ around 0 and find the radius of convergence. (4 Marks)

c) Let $f(z) = z^3$, $f : \mathbb{C} \to \mathbb{C}$. Determine real-valued functions u, v so that $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$). Verify that both u and v satisfy Laplace's equation. (5 Marks)

SECTION B (39 Marks)

Answer ANY THREE questions.

QUESTION THREE (13 Marks)

a) Let $w_0 \neq 0$ be a complex number such that $|w_0| = r$ and $\arg w_0 = \theta$. Find the polar forms of all the solutions z to $z^n = w_0$, where $n \geq 1$ is a positive integer. (4 Marks)

b) Let $z, w \in \mathbb{C}$. Show that

i) $\overline{z + w} = \bar{z} + \bar{w}$

ii) $\overline{zw} = \bar{z}\bar{w}$ (4 Marks)

c) Suppose that f is differentiable at z_0. Prove that f is continuous at z_0. (5 Marks)

QUESTION FOUR (13 Marks)

a) Derive formula for the real and imaginary parts of the following complex functions and check that they satisfy the Cauchy-Riemann equations:

i) $\sin z$ (3 Marks)

ii) $\cos z$ (3 Marks)

b) Show that, for $|z| < 1$, we have $f'(z) = \frac{\alpha f(z)}{1+z}$. (3 Marks)

c) Show that every polynomial p of degree at least 1 is surjective. (4 Marks)

QUESTION FIVE (13 Marks)

a) Find the values of

$$\int_{\gamma} x - y + ix^2 dz$$

where $z = x + iy$ and γ is:

i) the straight line joining 0 to $1 + i$. (3 Marks)

ii) the imaginary axis from 0 to i. (3 Marks)

iii) the line parallel to the real axis from i to $1 + i$. (3 Marks)

b) Find the zeros of the function $1 + e^x$. (4 Marks)

QUESTION SIX (13 Marks)

a) Let $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2 \sin z$ find an anti-derivative and calculate the integral along any smooth path from 0 to i. (5 marks)

b) Verify the associative law for multiplication of complex numbers. (4 Marks)

$$(z_1 z_2) z_3 = z_1 (z_2 z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

c) Suppose that z_1 and z_2 are complex numbers, with $z_1 z_2$ real and non-zero. Show that there exists a real number r such that $z_1 = r z_2$. (4 Marks)

QUESTION SEVEN (13 Marks)

a) Find all complex solutions of the following equations:

i) $\bar{z} = z$ (3 Marks)

ii) $\bar{z} + z = 0$ (3 Marks)

b) Verify that $\sqrt{2}|z| \geq |Re\ z| + |Im\ z|$. (3 Marks)

c) Express the all the 3rd roots of $-8i$ in the form $x + iy$, with $x, y \in \mathbb{R}$. (4 Marks)